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A SINGLE CELL HIGH ORDER SCHEME FOR THE CONVECTION–DIFFUSION EQUATION WITH VARIABLE COEFFICIENTS

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SUMMARY

A new finite difference scheme for the convection–diffusion equation with variable coefficients is proposed. The difference scheme is defined on a single square cell of size $2h$ over a 9-point stencil and has a truncation error of order h^4. The resulting system of equations can be solved by iterative methods. Numerical results of some test problems are given.

KEY WORDS Convection–Diffusion Finite Difference High Order

1. INTRODUCTION

In this paper we consider the convection–diffusion equation

$$Lu \equiv u_{xx} + u_{yy} + p(x, y)u_x + q(x, y)u_y = f(x, y) \tag{1}$$

This equation often appears in the description of transport phenomena. The magnitudes of $p(x, y)$ and $q(x, y)$ determine the ratio of the convection to diffusion. In many problems of practical interest the convective terms dominate the diffusion. Numerical simulation of (1) becomes increasingly difficult as the ratio of the convection to diffusion increases.

When the equation (1) is discretized using central differences, the resulting scheme, called the CDS, has a truncation error of order h^2. In the case of CDS, iterative methods for solving the resulting system of linear equations do not converge when the convective terms dominate and the cell Reynolds number is greater than a certain constant. In addition, direct methods for solving the system of linear equations may give erroneous results. If the convective terms are approximated by suitable forward or backward differences and the diffusion terms by central differences, the resulting scheme is called the 'upwind' or the 'upstream' scheme or the UDS. There are several variations of UDS and also combinations of the CDS and the UDS schemes. The UDS introduces artificial viscosity and hence the results are in error when the convection dominates. Both the truncation and the discretization error of UDS are of order h and hence a very fine mesh is needed if accurate results are required. Such a refinement of the mesh is often uneconomical.

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Recently, we proposed a new finite difference scheme for the special case of equation (1) where p and q are constants.[1] In this paper, we present a generalization of the scheme to the case of variable coefficients $p(x, y)$ and $q(x, y)$. The new scheme has a truncation error of order h^4 and the resulting system of linear equations can be solved by iterative methods even for large absolute values of $p(x, y)$ and $q(x, y)$.

The truncation error of our scheme, as given in the Appendix, is of order h^4. The coefficients of the non-diagonal terms in the difference equation do not have the same sign for all values of $p(x, y)$ and $q(x, y)$, therefore it is not possible to predict the order of the discretization error from that of the truncation error using the theoretical results known so far. There are some difference schemes for which the order of the discretization error is reduced by 1 as the transport number becomes large.[2] However, from the numerical results of several test problems, it appears that this is not the case for our method. The order of discretization error is defined for the asymptotic case when the mesh size $h \to 0$. The numerical estimates of the order determined from the errors calculated by using two different mesh sizes may not reach its asymptotic value as long as the derivatives appearing in the expression for the truncation error vary with the change in the mesh. This is indeed the case when the transport number is large. A better estimate of the order of the discretization error is obtained in such cases by refining the mesh. From the numerical results we conjecture that the order of our method is between 3 and 4.

We have solved several test problems using our scheme and also the UDS and the CDS. In almost all cases our scheme produced better results for a given mesh size. Only those schemes which are designed on the basis of the exact solution of a particular problem[3,4] give better results when the mesh is crude. Such schemes are a lot more complicated and difficult to implement. The rate at which the error decreases as the mesh is refined is not as fast as that of our scheme, and hence our scheme gives better results as the mesh is refined.

In the scheme proposed here, the cofficients can be computed easily when the grid is uniform. Our procedure can be extended to irregular meshes. In such cases, the difference scheme is not obtained explicity for each mesh point but is computed as the difference equations are assembled.[5] We have generalized our procedure to the case of the diffusion–convection equation when the diffusion coefficients are variable, and the resulting scheme has been applied to some problems of flows in porous media. The preliminary results of these extensions are quite promising and will be reported in the future. In this paper, we restrict our attention to the equation (1) and regular meshes.

In the derivation of the difference scheme, the solution $u(x, y)$ is first expressed locally on a mesh element in terms of a linear combination of the basis functions which are chosen to be polynomials in the present case. The functions $p(x, y)$, $q(x, y)$ and $f(x, y)$ are expanded in a similar manner. A set of linear equations for the unknown coefficients in the expansion of $u(x, y)$ are obtained by demanding that the differential equation (1) be satisfied locally. Additional equations are obtained by interpolating the solution over a set of mesh points which lie on the cell. This technique has been used to obtain single cell high order schemes for the Poisson, the Helmholtz, the biharmonic and other linear equations.[6,7]

The difference scheme derived here is a 9-point scheme. Only those mesh points which lie on a single square cell of side $2h$ are involved, thereby keeping the bandwidth as small as possible for the order of the truncation error achieved. No special formulae are needed for points near the boundary.

The new finite difference scheme for equation (1) is presented in the next section. Some details of its derivation are given in the Appendix. The results of numerical experiments with this scheme are given in Section 3.

2. THE FINITE DIFFERENCE SCHEME

The finite difference formula for a mesh point (x, y) which is denoted by '0' in Figure 1 involves the other eight mesh points at $(x \pm h, y)$, $(x, y \pm h)$, $(x \pm h, y \pm h)$. These points are denoted either by numbers 1–8 or by letters showing their directions with respect to the point '0' as in Figure 1. The difference formula involves the coefficients $\lambda_{i,j}$, $\mu_{i,j}$ and $c_{i,j}$ which appear in the expansions of $p(x, y)$, $q(x, y)$ and $f(x, y)$ along with the nodal values $u_k = u(x_k, y_k)$ for $k = 0, 1, 2, \ldots$. Alternatively, the coefficients in the expansion of the known functions can be expressed in terms of their partial derivatives. In practice it is more convenient to use the nodal values of the known functions rather than their derivatives. Therefore, an alternative formulation of the difference formula involving the nodal values of $p(x, y)$, $q(x, y)$ and $f(x, y)$ is also given.

The fourth order difference approximation of equation (1) is given by (see equation 19 in the Appendix)

$$\sum_{k=0}^{8} \alpha_k u_k = 6c_{0,0}h^2 + (c_{2,0} + c_{0,2})h^4 + (\lambda_{0,0}c_{1,0} + \mu_{0,0}c_{0,1})\frac{h^4}{2} \tag{2}$$

where

$$\begin{aligned}
\alpha_1 &\equiv \alpha_E = 4 + 2h\lambda_{0,0} + h^2R_3 + \frac{h^3}{2}R_6 \\
\alpha_2 &\equiv \alpha_N = 4 + 2h\mu_{0,0} + h^2R_4 + \frac{h^3}{2}R_5 \\
\alpha_3 &\equiv \alpha_W = 4 - 2h\lambda_{0,0} + h^2R_3 - \frac{h^3}{2}R_6 \\
\alpha_4 &\equiv \alpha_S = 4 - 2h\mu_{0,0} + h^2R_4 - \frac{h^3}{2}R_5 \\
\alpha_5 &\equiv \alpha_{NE} = 1 + \frac{h}{2}(\lambda_{0,0} + \mu_{0,0}) + \frac{h^2}{4}R_2 \\
\alpha_6 &\equiv \alpha_{NW} = 1 - \frac{h}{2}(\lambda_{0,0} - \mu_{0,0}) - \frac{h^2}{4}R_2 \\
\alpha_7 &\equiv \alpha_{SW} = 1 - \frac{h}{2}(\lambda_{0,0} + \mu_{0,0}) + \frac{h^2}{4}R_2 \\
\alpha_8 &\equiv \alpha_{SE} = 1 + \frac{h}{2}(\lambda_{0,0} - \mu_{0,0}) - \frac{h^2}{4}R_2 \\
\alpha_0 &\equiv -20 - h^2R_1
\end{aligned} \tag{3}$$

and where

$$\begin{aligned}
R_1 &= \lambda_{0,0}^2 + \mu_{0,0}^2 + 2\lambda_{1,0} + 2\mu_{0,1} \\
R_2 &= \mu_{1,0} + \lambda_{0,1} + \lambda_{0,0}\mu_{0,0} \\
R_3 &= \tfrac{1}{2}\lambda_{0,0}^2 + \lambda_{1,0} \\
R_4 &= \tfrac{1}{2}\mu_{0,0}^2 + \mu_{0,1} \\
R_5 &= \mu_{2,0} + \mu_{0,2} + \tfrac{1}{2}\lambda_{0,0}\mu_{1,0} + \tfrac{1}{2}\mu_{0,0}\mu_{0,1} \\
R_6 &= \lambda_{2,0} + \lambda_{0,2} + \tfrac{1}{2}\lambda_{0,0}\lambda_{1,0} + \tfrac{1}{2}\mu_{0,0}\lambda_{0,1}
\end{aligned} \tag{4}$$

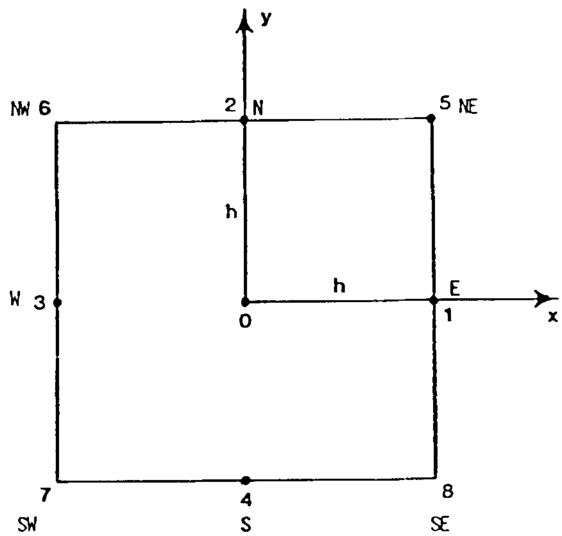

Figure 1. Labelling of grid points

Replacing $\lambda_{i,j}$, $\mu_{i,j}$ and $c_{i,j}$ in terms of the nodal values of the functions $p(x, y)$, $q(x, y)$ and $f(x, y)$ we get an alternative finite difference scheme given by

$$\sum_{j=0}^{8} \alpha_j u_j = \frac{h^2}{2}[f_N + f_S + f_E + f_W + 8f_0] + \frac{h^3}{4}[p_0(f_E - f_W)] + q_0(f_N - f_S)] \qquad (5)$$

where

$$\alpha_1 \equiv \alpha_E = 4 + \frac{h}{4}[4p_0 + 3p_E - p_W + p_N + p_S] + \frac{h^2}{8}[4p_0^2 + p_0(p_E - p_W) + q_0(p_N - p_S)]$$

$$\alpha_2 \equiv \alpha_N = 4 + \frac{h}{4}[4q_0 + 3q_N - q_S + q_E + q_W] + \frac{h^2}{8}[4q_0^2 + p_0(q_E - q_W) + q_0(q_N - q_S)]$$

$$\alpha_3 \equiv \alpha_W = 4 - \frac{h}{4}[4p_0 - p_E + 3p_W + p_N + p_S] + \frac{h^2}{8}[4p_0^2 - p_0(p_E - p_W) - q_0(p_N - p_S)]$$

$$\alpha_4 \equiv \alpha_S = 4 - \frac{h}{4}[4q_0 - q_N + 3q_S + q_E + q_W] + \frac{h^2}{8}[4q_0^2 - p_0(q_E - q_W) - q_0(q_N - q_S)]$$

$$\alpha_5 \equiv \alpha_{NE} = 1 + \frac{h}{2}(p_0 + q_0) + R_7$$

$$\alpha_6 \equiv \alpha_{NW} = 1 - \frac{h}{2}(p_0 - q_0) - R_7$$

$$\alpha_7 \equiv \alpha_{SW} = 1 - \frac{h}{2}(p_0 + q_0) + R_7$$

$$\alpha_8 \equiv \alpha_{SE} = 1 + \frac{h}{2}(p_0 - q_0) - R_7$$

$$\alpha_0 = -[20 + h^2(p_0^2 + q_0^2) + h(p_E - p_W) + h(q_N - q_S)]$$

and where

$$R_7 = \frac{h}{8}(q_E - q_W + p_N - p_S) + \frac{h^2}{4} p_0 q_0$$

Note that both the difference schemes (2) and (5) reduce to the scheme given in Reference 1 when $p(x, y)$ and $q(x, y)$ are constants.

3. NUMERICAL RESULTS

Numerical results for several test problems have been obtained by using the two 9-point schemes given in (2) and (5). The scheme (2) involves the derivatives of the functions $p(x, y)$, $q(x, y)$ and $f(x, y)$ whereas the scheme (5) involves only the nodal values of these functions. The results obtained by these two schemes do not differ significantly as far as the order of the maximum errors are concerned. Hence the results obtained by using the scheme (5), which is more useful in practice, are reported here. This scheme is called the single cell high order scheme or SCHOS in the sequel.

All test problems given here are solved on a unit square $[0, 1]\times[0, 1]$ using a uniform mesh h. Boundary values of the solutions are assumed to be known. The system of linear equations is solved by using the successive over-relaxation (SOR) iterative method. Sometimes it is necessary to use a relaxation parameter less than 1 with the CDS. The convergence criterion for the iteration was chosen to be 10^{-6}. All the calculations were done in single precision on a Dec 20 or an IBM 4341.

Test problem 1[3]

Consider the boundary value problem

$$\left.\begin{aligned} &-\varepsilon(u_{xx}+u_{yy})+u_x=0, && && 0\le x, y\le 1\\ &u(x,0)=0, && u(x,1)=0, && 0\le x\le 1\\ &u(0,y)=\sin \pi y, && u(1,y)=2\sin \pi y, && 0\le y\le 1 \end{aligned}\right\} \quad (6)$$

Comparison of (6) and (1) shows that $-p(x, y) = 1/\varepsilon = P(\text{say})$, $q(x, y)\equiv 0$ and $f(x, y)\equiv 0$. The exact solution

$$u = e^{Px/2} \sin \pi y[2e^{-P/2} \sinh \sigma x + \sinh \sigma(1-x)]/\sinh \sigma$$

where $\sigma^2 = \pi^2 + P^2/4$, shows the presence of a boundary layer near $x = 1$ whose thickness is of order $1/P$ for P large. The boundary layer is expected to affect the numerical results adversely as P increases.

This problem has been studied by Gartland,[3] who proposed a special five point stencil involving modified Bessel functions. In Tables I–III we give some results for this problem.

Table I. Maximum relative errors for problem 1, $h = 1/32$

P	UDS	CDS	SCHOS
10	0·9166(−1)	0·4537(−2)	0·6011(−4)
20	0·1262	0·1576(−1)	0·1399(−3)
40	0·1686	0·5925(−1)	0·1511(−2)
100	0·2264	0·3002	0·3517(−1)

Table II. Maximum absolute error and the estimated orders for problem 1

P	h^{-1}	CDS	Order	SCHOS	Order
40	8	0·5122		0·1256	
	16	0·2310	1·15	0·2009(−1)	2·65
	32	0·6723(−1)	1·78	0·1712(−2)	3·55
100	8	0·9060		0·4249	
	16	0·5618	0·68	0·1670	1·35
	32	0·2872	0·97	0·3365(−1)	2·31
	64	0·9493(−1)	1·60	0·3151(−2)	3·41

Table III. Average relative errors for problem 1, $P=100$

h^{-1}	CDS	SCHOS	Gartland
8	1·18(1)	8·13(−2)	6·81(−3)
16	1·46(−1)	1·25(−2)	4·94(−3)
32	1·61(−2)	1·26(−3)	1·90(−3)
64	2·23(−3)	1·79(−4)	5·71(−4)

In Table I the maximum relative errors for $P=10$, 20, 40 and 100 for $h=1/32$ show that the errors due to UDS range between 9 and 23 per cent, whereas the CDS gives acceptable results for $P\leq 40$ but not for $P=100$. The errors for SCHOS remain ≤4 per cent. As expected, the maximum relative errors occur near the corners at $x=1$ in all cases.

In Table II, the maximum absolute errors for different values of h are given. Numerical estimates for the order of the discretization errors which are obtained by considering the errors due to mesh sizes h and $2h$ are also given. It is clear from the table that the order of SCHOS is about twice that of CDS.

In order to compare our results with those given by Gartland, we give the average relative errors for $P=100$ in Table III. Clearly, the special method proposed by Gartland gives better results when the mesh is crude. However, as the mesh is refined, the errors due to SCHOS decrease rapidly and for $h\leq 1/32$ the errors due to SCHOS are consistently smaller. The order of the Gartland method is not even as high as that of the CDS.

Gartland proposed a boundary correction stencil to be used at mesh points on $x=1-h$. Once this correction is done, the results obtained by CDS, SCHOS and the Gartland scheme are comparable for $h=1/8$. However, as the mesh is refined, the results given by SCHOS are better than those given by the other two methods.

Test problem 2[4]

Consider the boundary value problem

$$\begin{aligned} \phi_{xx}+\phi_{yy} &= P\cos\theta\phi_x + P\sin\theta\phi_y, && 0\leq x,\ y\leq 1\\ \phi(x,0)&=0,\ \phi(x,1)=0, && 0\leq x\leq 1 \\ \phi(0,y)&=4y(1-y),\ \phi(1,y)=0, && 0\leq y\leq 1 \end{aligned} \qquad (7)$$

Table IV. Maximum absolute errors and the orders for problem 2

	h^{-1}	UDS	Order	CDS	Order	SCHOS	Order
$\theta=0$	8	0·1227		0·3420		0·8280(−1)	
	16	0·1604		0·1532	1·15	0·1323(−1)	2·65
	32	0·1256	0·35	0·4449(−1)	1·78	0·1123(−2)	3·55
$\theta=\pi/8$	8	0·2886		0·3688		0·6931(−1)	
	16	0·2268	0·34	0·1286	1·52	0·1019(−1)	2·77
	32	0·1394	0·70	0·3484(−1)	1·88	0·8128(−3)	3·64
$\theta=\pi/4$	8	0·2467		0·2833		0·4932(−1)	
	16	0·2035	0·28	0·8031(−1)	1·82	0·5977(−2)	3·04
	32	0·1218	0·74	0·1950(−1)	2·04	0·4066(−3)	3·88

Comparison of (1) and (7) shows that $p(x, y)=-P\cos\theta$ and $q(x, y)=-P\sin\theta$ and $f(x, y)=0$. The exact solution is given by

$$\phi=e^{-P(x\cos\theta+y\sin\theta)/2}\sum_{n=1}^{\infty}B_n\sinh[\sigma_n(1-x)]\sin n\pi y$$

where

$$\sigma_n^2=n^2\pi^2+P^2/4$$

and

$$B_n=\frac{8}{\sin h\sigma_n}\int_0^1 y(1-y)e^{-p\sin\theta y/2}\sin n\pi y\,dy$$

This problem represents the convection of θ (temperature or concentration) in a fluid moving with a uniform velocity at an angle θ to the x-axis. For $\theta=0$ a boundary layer develops on $x=1$ as in the case of problem 1, whereas for $\theta\neq0$, boundary layers develop on $x=1$ and also $y=1$ for P large. Numerical results obtained by using UDS are known to be affected adversely when the direction of the flow is not aligned with the direction of the finite difference grid. This is known as the grid orientation problem. We have chosen this problem to study whether the numerical solutions obtained by SCHOS are affected by the grid orientation. This problem has been studied by Stubley, Raithby and Strong in a recent paper[4] in which they proposed a special method called QIS which uses a 9-point stencil.

In Table IV we give the absolute maximum errors for $P=40$, $h=1/8$, 1/16 and 1/32 for $\theta=0$, $\pi/8$ and $\pi/4$. Estimates of the order of the method obtained from the numerical results are also given. Most of the observations made in the case of problem 1 remain valid for this problem as well. It is clearly seen that the results of UDS deteriorate as θ increases, whereas this is not the case for both the CDS and the SCHOS. The results for $\theta=\pi/4$ appear to be better than those for $\theta=0$, however, this is due to a decrease in the effective transport number for $\theta=\pi/4$. In any case the SCHOS is not affected by the grid orientation. As in the case of problem 1, the results obtained by using special methods such as QIS for this problem are more accurate than those obtained by SCHOS when the mesh is crude. However, as the mesh is refined, the results obtained by SCHOS improve rapidly. The maximum error over a coarse 7×7 mesh ($h=1/8$) for $P=80$ are given in Reference 4. For $h=1/32$ the results given by SCHOS and QIS are comparable, whereas for $h=1/8$ and 1/16, QIS results are better.

Table V. Maximum absolute errors for problem 3, equation (8)

P	h^{-1}	UDS	Order	CDS	Order	SCHOS	Order
100	8	0·1678		0·6174(−2)		0·3081(−2)	
	16	0·9894(−1)	0·76	0·1633(−2)	1·92	0·2615(−3)	3·56
	32	0·5426(−1)	0·86	0·4154(−3)	1·97	0·1775(−4)	3·88
1000	8	0·2017		—		0·9448(−2)	
	16	0·1238	0·70	—		0·1845(−2)	2·36
	32	0·6819(−1)	0·86	—		0·1864(−3)	3·31

Test problem 3

In problems 1 and 2 the coefficient functions were constants. Now we consider a problem where these functions are variables; thus in (1) let

$$\left.\begin{aligned} p(x, y) = Px, \qquad q(x, y) = -Py \\ \text{with the exact solution} \qquad \\ u = xy(1-x)(1-y)e^{x+y} \end{aligned}\right\} \tag{8}$$

Numerical solutions for $P = 100$ and 1000 are given in Table V. For $P = 1000$ the CDS failed to converge with S.O.R.

Instead of considering $p(x, y)$ and $q(x, y)$ as first degree polynomials, we considered another problem with the exact solution the same as in (8) but $p(x, y) = \exp(x+y)$ and $q(x, y) = 1/\exp(x+y)$. Here again the results were similar to those given for (8). The CDS did not converge for $P > 100$.

4. CONCLUSIONS

Several test problems have been solved using the single cell high order method proposed here. From the numerical experiments it appears that the scheme gives good results. Further testing over a wider range of the values of the parameters is necessary before its usefulness in practice is established. The scheme is simple, easy to implement and the resulting system of linear equations can be solved by iterative methods. The rate of convergence of our scheme is twice that of the central difference scheme and about 3 to 4 times that of the upwind difference scheme. The proposed scheme is not affected by the grid orientation nor does it introduce artificial viscosity. The method of derivation carried over to irregular meshes with some modifications.

The difference scheme has been arrived at by approximating the solution locally by means of polynomials. Therefore, the accuracy of the numerical solutions will be affected for mesh sizes for which such an approximation is not sufficiently accurate. In particular when the convection is large compared with the diffusion and a boundary layer exists in which the solution varies exponentially, one could expect a deterioration of the numerical results if a crude mesh is used. This can be seen in problems 1 and 2 for the values of $Ph > 6$.

In the derivation of the scheme it is assumed that the solution of the problem is sufficiently smooth. If the smoothness condition is not satisfied, the order of the scheme drops and it may give results no better than the lower order schemes.

The proposed scheme uses a 9-point stencil and hence requires additional computations as compared to the 5-point schemes. In order to compare methods for a given order of the error it is necessary to obtain work estimates in terms of arithmetic operations. Of course the rate of convergence of the iterative method also plays a decisive role in such estimates. Comparison of such estimates is under investigation.

APPENDIX. DERIVATION OF THE SINGLE CELL HIGH ORDER DIFFERENCE SCHEMES FOR CONVECTION-DIFFUSION EQUATION

The differential equation is

$$u_{xx}+u_{yy}+p(x,y)u_x+q(x,y)u_y=f(x,y) \tag{9}$$

We assume that, locally, the solution $u(x, y)$ and the functions p, q, f can be expressed by two-dimensional power series:

$$\begin{aligned} u(x,y)&=\sum a_{i,j}x^iy^j, \qquad & f(x,y)&=\sum c_{i,j}x^iy^j \\ p(x,y)&=\sum \lambda_{i,j}x^iy^j, \qquad & q(x,y)&=\sum \mu_{i,j}x^iy^j \end{aligned} \tag{10}$$

Substituting (10) into (9) and comparing the coefficients of x^iy^j, we get

$$\begin{aligned} c_{i,j}&=(i+1)(i+2)a_{i+2,j}+(j+1)(j+2)a_{i,j+2} \\ &\quad+\sum_{i\le r,s\le j}[(i-r+1)\lambda_{r,s}a_{i+1-r,j-s}+(j-s+1)\mu_{r,s}a_{i-r,j+1-s}] \end{aligned} \tag{11}$$

The equations (11) constitute the constraints imposed by the differential equation (9) on the coefficients $a_{i,j}$, $c_{i,j}$, $\lambda_{i,j}$ and $\mu_{i,j}$ of the expansions (10). In particular,

$$\begin{aligned} c_{0,0}&=2(a_{2,0}+a_{0,2})+\lambda_{0,0}a_{1,0}+\mu_{0,0}a_{0,1} \\ c_{1,0}&=6a_{3,0}+2a_{1,2}+2\lambda_{0,0}a_{2,0}+\lambda_{1,0}a_{1,0}+\mu_{0,0}a_{1,1}+\mu_{1,0}a_{0,1} \\ c_{0,1}&=2a_{2,1}+6a_{0,3}+\lambda_{0,0}a_{1,1}+\lambda_{0,1}a_{1,0}+2\mu_{0,0}a_{0,2}+\mu_{0,1}a_{0,1} \\ c_{2,0}&=12a_{4,0}+2a_{2,2}+3\lambda_{0,0}a_{3,0}+2\lambda_{1,0}a_{2,0}+\lambda_{2,0}a_{1,0}+\mu_{0,0}a_{2,1}+\mu_{1,0}a_{1,1}+\mu_{2,0}a_{0,1} \\ c_{0,2}&=2a_{2,2}+12a_{0,4}+\lambda_{0,0}a_{1,2}+\lambda_{0,1}a_{1,1}+\lambda_{0,2}a_{1,0}+3\mu_{0,0}a_{0,3}+2\mu_{0,1}a_{0,2}+\mu_{0,2}a_{0,1} \\ c_{1,1}&=6a_{3,1}+6a_{1,3}+2\lambda_{0,0}a_{2,1}+\lambda_{1,0}a_{1,1}+\lambda_{1,1}a_{1,0}+2\mu_{0,0}a_{1,2}+\mu_{0,1}a_{1,1}+\mu_{1,1}a_{0,1} \end{aligned} \tag{12}$$

The above six constraints ensure the satisfaction of the differential equation (9) for $u=x^iy^j$ for $i+j\le 4$. The constraints involve 15 unknown values of $a_{i,j}$, $0\le i+j\le 4$. The remaining nine equations, relating the values of $a_{i,j}$, are obtained by collocation on the nine points 0–8 of the single cell of side $2h$ (see Figure 1). In particular,

$$\begin{aligned} u_N&\equiv u_2=a_{0,0}+a_{0,1}h+a_{0,2}h^2+a_{0,3}h^3+a_{0,4}h^4+\dots \\ u_{NE}&\equiv u_5=a_{0,0}+(a_{1,0}+a_{0,1})h+(a_{2,0}+a_{1,1}+a_{0,2})h^2+\dots \end{aligned} \tag{13}$$

We use the notation

$$\begin{aligned} \Diamond u_0&=u_1+u_2+u_3+u_4 \\ \Box u_0&=u_5+u_6+u_7+u_8 \end{aligned} \tag{14}$$

From (13) we obtain relations of the form

$$\begin{aligned} u_1 - u_0 &= a_{1,0}h + a_{2,0}h^2 + \ldots \\ u_2 + u_4 - 2u_0 &= 2a_{0,2}h^2 + 2a_{0,4}h^4 + \ldots \\ u_5 + u_7 - u_6 - u_8 &= 4a_{1,1}h^2 + O(h^4) \end{aligned} \tag{15}$$

Now, from (12) we have,

$$\begin{aligned} h^2 c_{0,0} &= 2h^2(a_{2,0} + a_{0,2}) + \lambda_{0,0}h(a_{1,0}h) + \mu_{0,0}h(a_{0,1}h) \\ &= 2h^2(a_{2,0} + a_{0,2}) + \lambda_{0,0}h(u_1 - u_0 - a_{2,0}h^2 - a_{3,0}h^3 \ldots) \\ &\quad + \mu_{0,0}h(u_2 - u_0 - a_{0,2}h^2 - a_{0,3}h^3 \ldots) \\ &= (\Diamond u_0 - 4u_0 - 2a_{4,0}h^4 - 2a_{0,4}h^4) + \lambda_{0,0}h(u_1 - u_0 - a_{2,0}h^2) \\ &\quad + \mu_{0,0}h(u_2 - u_0 - a_{0,2}h^2) + O(h^4) \end{aligned} \tag{16}$$

Neglecting terms of order h^3, we obtain the well known upwind difference scheme of order h.

$$h^2 c_{0,0} = \Diamond u_0 - 4u_0 + \lambda_{0,0}h(u_1 - u_0) + \mu_{0,0}h(u_2 - u_0) \tag{17}$$

If the terms containing h^3 are eliminated in (16), we obtain

$$\begin{aligned} h^2 c_{0,0} &= \Diamond u_0 - 4u_0 + \lambda_{0,0}h(u_1 - u_0) + \mu_{0,0}h(u_2 - u_0) \\ &\quad - \frac{\lambda_{0,0}h}{2}(u_1 + u_3 - 2u_0 + O(h^4)) - \frac{\mu_{0,0}h}{2}(u_2 + u_4 - 2u_0 + O(h^4)) \\ &= \Diamond u_0 - 4u_0 + \lambda_{0,0}\frac{h}{2}(u_1 - u_3) + \mu_{0,0}\frac{h}{2}(u_2 - u_4) + O(h^4) \end{aligned} \tag{18}$$

The central difference scheme results when $O(h^4)$ terms are neglected in (18). This result cannot be improved any further without using more constraints from (11), (12). The two constraints involving $c_{1,0}$, $c_{0,1}$ contain $a_{3,0}$, $a_{0,3}$ which also appear in the next set of constraints for $c_{2,0}$, $c_{0,2}$. Using these constraints and relations in (13), (14), (15) we obtain,

$$\begin{aligned} &6h^2 c_{0,0} + h^4/2(c_{1,0}\lambda_{0,0} + c_{0,1}\mu_{0,0}) + h^4(c_{2,0} + c_{0,2}) \\ &\quad = 6\Diamond u_0 - 24u_0 + 3h\lambda_{0,0}(u_1 - u_3) + 3h\mu_{0,0}(u_2 - u_4) \\ &\qquad + 2h^4(\lambda_{0,0}a_{1,2} + \mu_{0,0}a_{2,1}) + a_{2,0}h^4(\lambda_{0,0}^2 + 2\lambda_{1,0}) \\ &\qquad + a_{0,2}h^4(\mu_{0,0}^2 + 2\mu_{0,1}) + a_{1,1}h^4(\lambda_{0,0}\mu_{0,0} + \lambda_{0,1} + \mu_{1,0}) \\ &\qquad + a_{1,0}h^4(\lambda_{2,0} + \lambda_{0,2} + \tfrac{1}{2}\lambda_{0,0}\lambda_{1,0} + \tfrac{1}{2}\mu_{0,0}\lambda_{0,1}) \\ &\qquad + a_{0,1}h^4(\mu_{2,0} + \mu_{0,2} + \tfrac{1}{2}\lambda_{0,0}\mu_{1,0} + \tfrac{1}{2}\mu_{0,0}\mu_{0,1}) + 4h^4 a_{2,2} + O(h^6) \\ &\quad = 4\Diamond u_0 + \Box u_0 - 20u_0 + 2h\lambda_{0,0}(u_1 - u_3) + 2h\mu_{0,0}(u_2 - u_4) \\ &\qquad + \lambda_{0,0}h/2(u_5 - u_6 - u_7 + u_8) + \mu_{0,0}h/2(u_5 + u_6 - u_7 - u_8) \\ &\qquad + \tfrac{1}{2}h^2(\lambda_{0,0}^2 + 2\lambda_{1,0})(u_1 + u_3) + \tfrac{1}{2}h^2(\mu_{0,0}^2 + 2\mu_{0,1})(u_2 + u_4) \\ &\qquad - h^2(\lambda_{0,0}^2 + \mu_{0,0}^2 + 2\lambda_{1,0} + 2\mu_{0,1})u_0 + \tfrac{1}{4}h^2(\lambda_{0,0}\mu_{0,0} + \lambda_{0,1} + \mu_{1,0})(u_5 - u_6 + u_7 - u_8) \\ &\qquad + h^3/2(\lambda_{2,0} + \lambda_{0,2} + \tfrac{1}{2}\lambda_{0,0}\lambda_{1,0} + \tfrac{1}{2}\mu_{0,0}\lambda_{0,1})(u_1 - u_3) \\ &\qquad + h^3/2(\mu_{2,0} + \mu_{0,2} + \tfrac{1}{2}\lambda_{0,0}\mu_{1,0} + \tfrac{1}{2}\mu_{0,0}\mu_{0,1})(u_2 - u_4) + O(h^6) \end{aligned} \tag{19}$$

Neglecting the terms of order h^6 we get the fourth order difference scheme given by equation (2). The truncation error of this scheme is given by $\frac{1}{6}h^4\psi(x, y)$ where $\psi(x, y)$

represents the coefficient of h^6 on the right hand side of (19). The function $\psi(x, y)$ is given by

$$\begin{aligned}\psi(x, y) = {} & 12(a_{6,0} + a_{0,6}) + 6(\lambda_{0,0}a_{5,0} + \mu_{0,0}a_{0,5}) + 4(a_{4,2} + a_{2,4}) \\ & + 2\lambda_{0,0}(a_{3,2} + a_{1,4}) + 2\mu_{0,0}(a_{4,1} + a_{2,3}) + a_{4,0}(\lambda_{0,0}^2 + 2\lambda_{1,0}) \\ & + a_{0,4}(\mu_{0,0}^2 + 2\mu_{0,1}) + (a_{3,1} + a_{1,3})(\lambda_{0,0}\mu_{0,0} + \lambda_{0,1} + \mu_{1,0}) \\ & + (\lambda_{2,0} + \lambda_{0,2} + \tfrac{1}{2}\lambda_{0,0}\mu_{1,0} + \tfrac{1}{2}\mu_{0,0}\lambda_{0,1})a_{3,0} \\ & + (\mu_{2,0} + \mu_{0,2} + \tfrac{1}{2}\lambda_{0,0}\mu_{1,0} + \tfrac{1}{2}\mu_{0,0}\mu_{0,1})a_{0,3} \end{aligned} \tag{20}$$

In the above expressions, $a_{i,j}$, $\lambda_{i,j}$ and $\mu_{i,j}$ are defined as follows:

$$a_{i,j} = \frac{1}{i!\,j!}\frac{\partial^{i+j}u}{\partial x^i\,\partial y^j}, \qquad \lambda_{i,j} = \frac{1}{i!\,j!}\frac{\partial^{i+j}p}{\partial x^i\,\partial y^j}, \qquad \mu_{i,j} = \frac{1}{i!\,j!}\frac{\partial^{i+j}q}{\partial x^i\,\partial y^j} \tag{21}$$

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